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Approximation by Nörlund Means of Walsh–Fourier Series

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We study the rate of approximation by Nörlund means for Walsh–Fourier series of a function in L^p and, in particular, in Lip(α, p) over the unit interval $[0, 1)$, where $\alpha > 0$ and $1 \leqslant p \leqslant \infty$. In case $p = \infty$, by L^p we mean C_W, the collection of the uniformly W-continuous functions over $[0, 1)$. As special cases, we obtain the earlier results by Yano, Jastrebova, and Skvorcov on the rate of approximation by Cesàro means. Our basic observation is that the Nörlund kernel is quasi-positive, under fairly general assumptions. This is a consequence of a Sidon type inequality. At the end, we raise two problems.

1. INTRODUCTION

We consider the Walsh orthonormal system $\{w_k(x): k \geqslant 0\}$ defined on the unit interval $I = [0, 1)$ in the Paley enumeration (see [4]). To be more specific, let

$$r_0(x) := \begin{cases} 1 & \text{if } x \in [0, 2^{-1}), \\ -1 & \text{if } x \in [2^{-1}, 1), \end{cases}$$

$$r_0(x+1) := r(x),$$

$$r_j(x) := r_0(2^j x), \qquad j \geqslant 1 \text{ and } x \in I,$$

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be the well-known Rademacher functions. For $k=0$ set $w_0(x)=1$, and if

$$k := \sum_{j=0}^{\infty} k_j 2^j, \qquad k_j = 0 \text{ or } 1,$$

is the dyadic representation of an integer $k \geqslant 1$, then set

$$w_k(x) := \prod_{j=0}^{\infty} [r_j(x)]^{k_j}. \tag{1.1}$$

We denote by $\mathscr{P}_n$ the collection of Walsh polynomials of order less than n, that is, functions of the form

$$P(x) := \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \geqslant 1$ and $\{a_k\}$ is any sequence of real (or complex) numbers.

Denote by Σ_m the finite σ-algebra generated by the collection of dyadic intervals of the form

$$I_m(k) := [k2^{-m}, (k+1)\, 2^{-m}), \qquad k = 0, 1, ..., 2^m - 1,$$

where $m \geqslant 0$. It is not difficult to see that the collection of Σ_m-measurable functions on I coincides with $\mathscr{P}_{2^m}$, $m \geqslant 0$.

We will study approximation by means of Walsh polynomials in the norm of $L^p = L^p(I)$, $1 \leqslant p < \infty$, and $C_W = C_W(I)$. We remind the reader that C_W is the collection of functions $f: I \to \mathbf{R}$ that are uniformly continuous from the dyadic topology of I to the usual topology of $\mathbf{R}$, or in short, uniformly W-continuous. The dyadic topology is generated by the union of Σ_m for $m = 0, 1, ...$.

As is known (see, e.g., [6, p. 9]), a function belongs to C_W if and only if it is continuous at every dyadic irrational of I, is continuous from the right on I, and has a finite limit from the left on $(0, 1]$, all these in the usual topology. Hence it follows immediately that if the periodic extension of a function f from I to $\mathbf{R}$ with period 1 is classically continuous, then f is also uniformly W-continuous on I. The converse statement is not true. For example, the Walsh functions w_k belong to C_W, but they are not classically continuous for $k \geqslant 1$.

For the sake of brevity in notation, we agree to write L^∞ instead of C_W and set

$$\|f\|_p := \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|f\|_\infty := \sup\{|f(x)| : x \in I\}.$$

After these preliminaries, the best approximation of a function $f \in L^p$, $1 \leqslant p \leqslant \infty$, by polynomials in $\mathscr{P}_n$ is defined by

$$E_n(f, L^p) := \inf_{P \in \mathscr{P}_n} \|f - P\|_p.$$

Since $\mathscr{P}_n$ is a finite dimensional subspace of L^p for any $1 \leqslant p \leqslant \infty$, this infimum is attained.

From the results of [6, pp. 142 and 156–158] it follows that L^p is the closure of the Walsh polynomials when using the norm $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$. In particular, C_W is the uniform closure of the Walsh polynomials.

Next, define the modulus of continuity in L^p, $1 \leqslant p \leqslant \infty$, of a function $f \in L^p$ by

$$\omega_p(f, \delta) := \sup_{|t| < \delta} \|\tau_t f - f\|_p, \qquad \delta > 0,$$

where τ_t means dyadic translation by t:

$$\tau_t f(x) := f(x \dotplus t), \qquad x, t \in I.$$

Finally, for each $\alpha > 0$, Lipschitz classes in L^p are defined by

$$\operatorname{Lip}(\alpha, p) := \{f \in L^p : \omega_p(f, \delta) = \mathscr{O}(\delta^\alpha) \text{ as } \delta \to 0\}.$$

Unlike the classical case, $\operatorname{Lip}(\alpha, p)$ is not trivial when $\alpha > 1$. For example, the function $f := w_0 + w_1$ belongs to $\operatorname{Lip}(\alpha, p)$ for all $\alpha > 0$ since

$$\omega_p(f, \delta) = 0 \qquad \text{when} \quad 0 < \delta < 2^{-1}.$$

2. Main Results

Given a function $f \in L^1$, its Walsh–Fourier series is defined by

$$\sum_{k=0}^{\infty} a_k w_k(x), \qquad \text{where} \quad a_k := \int_0^1 f(t)\, w_k(t)\, dt. \tag{2.1}$$

The nth partial sums of series in (2.1) are

$$s_n(f, x) := \sum_{k=0}^{n-1} a_k w_k(x), \qquad n \geqslant 1.$$

As is well known,

$$s_n(f, x) = \int_0^1 f(x \dotplus t)\, D_n(t)\, dt,$$

where

$$D_n(t) := \sum_{k=0}^{n-1} w_k(t), \qquad n \geqslant 1,$$

is the Walsh–Dirichlet kernel of order n.

Let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers. The Nörlund means for series (2.1) are defined by

$$t_n(f, x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} s_k(f, x),$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k, \qquad n \geqslant 1.$$

We always assume that $q_0 > 0$ and

$$\lim_{n \to \infty} Q_n = \infty. \tag{2.2}$$

In this case, the summability method generated by $\{q_k\}$ is regular if and only if

$$\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0. \tag{2.3}$$

As to this notion and result, we refer the reader to [2, pp. 37–38].

We note that in the particular case when $q_k = 1$ for all k, these $t_n(f, x)$ are the first arithmetic or $(C, 1)$-means. More generally, when

$$q_k = A_k^\beta := \binom{\beta + k}{k} \qquad \text{for} \quad k \geqslant 1 \text{ and } q_0 = A_0^\beta := 1,$$

where $\beta \neq -1, -2, \ldots$, the $t_n(f, x)$ are the (C, β)-means for series (2.1).

The representation

$$t_n(f, x) = \int_0^1 f(x \dotplus t)\, L_n(t)\, dt \tag{2.4}$$

plays a central role in the sequel, where

$$L_n(t) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k(t), \qquad n \geqslant 1, \tag{2.5}$$

is the so-called Nörlund kernel.

Our main results read as follow.

THEOREM 1. *Let* $f \in L^p$, $1 \leqslant p \leqslant \infty$, *let* $n = 2^m + k$, $1 \leqslant k \leqslant 2^m$, $m \geqslant 1$, *and let* $\{q_k : k \geqslant 0\}$ *be a sequence of nonnegative numbers such that*

$$\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = \mathcal{O}(1) \qquad \text{for some} \quad 1 < \gamma \leqslant 2. \tag{2.6}$$

If $\{q_k\}$ *is nondecreasing, then*

$$\|t_n(f) - f\|_p \leqslant \frac{5}{2Q_n} \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}) + \mathcal{O}\{\omega_p(f, 2^{-m})\}, \tag{2.7}$$

while if $\{q_k\}$ *is nonincreasing, then*

$$\|t_n(f) - f\|_p \leqslant \frac{5}{2Q_n} \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, \omega_p(f, 2^{-j}) + \mathcal{O}\{\omega_p(f, 2^{-m})\}. \tag{2.8}$$

Clearly, condition (2.6) implies (2.2) and (2.3).

We note that if $\{q_k\}$ is nondecreasing, in sign $q_k \uparrow$, then

$$\frac{nq_{n-1}}{Q_n} = \mathcal{O}(1) \tag{2.9}$$

is a sufficient condition for (2.6). In particular, (2.9) is satisfied if

$$q_k \asymp k^{\beta} \text{ or } (\log k)^{\beta} \qquad \text{for some} \quad \beta > 0.$$

Here and in the sequel, $q_k \asymp r_k$ means that the two sequences $\{q_k\}$ and $\{r_k\}$ have the same order of magnitude; that is, there exist two positive constants C_1 and C_2 such that

$$C_1 r_k \leqslant q_k \leqslant C_2 r_k \qquad \text{for all } k \text{ large enough.}$$

If $\{q_k\}$ is nonincreasing, in sign $q_k \downarrow$, then condition (2.6) is satisfied if, for example,

$$\begin{aligned} &\text{(i)} \quad q_k \asymp k^{-\beta} && \text{for some} \quad 0 < \beta < 1, \text{ or} \\ &\text{(ii)} \quad q_k \asymp (\log k)^{-\beta} && \text{for some} \quad 0 < \beta. \end{aligned} \tag{2.10}$$

Namely, it is enough to choose $1 < \gamma < \min(2, \beta^{-1})$ in case (i), and $\gamma = 2$ in case (ii).

THEOREM 2. *Let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers such that in case $q_k \uparrow$ condition* (2.9) *is satisfied, while in case $q_k \downarrow$ condition* (2.10) *is satisfied. If $f \in \operatorname{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leqslant p \leqslant \infty$, then*

$$\|t_n(f) - f\|_p = \begin{cases} \mathcal{O}(n^{-\alpha}) & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-1} \log n) & \text{if } \alpha = 1, \\ \mathcal{O}(n^{-1}) & \text{if } \alpha > 1. \end{cases} \tag{2.11}$$

Now we make a few historical comments. The rate of convergence of (C, β)-means for functions in $\operatorname{Lip}(\alpha, p)$ was first studied by Yano [10] in the cases when $0 < \alpha < 1$, $\beta > \alpha$, and $1 \leqslant p \leqslant \infty$; then by Jastrebova [1] in the case when $\alpha = \beta = 1$ and $p = \infty$. Later on, Skvorcov [7] showed that these estimates hold for $0 < \beta \leqslant \alpha$ as well, and also studied the cases when $\alpha = 1$, $\beta > 0$, and $1 \leqslant p \leqslant \infty$. In their proofs, the above authors rely heavily on the specific properties of the binomial coefficients A_k^β.

Watari [8] proved that a function $f \in L^p$ belongs to $\operatorname{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leqslant p \leqslant \infty$ if and only if

$$E_n(f, L^p) = \mathcal{O}(n^{-\alpha}).$$

Thus, for $0 < \alpha < 1$ the rate of approximation to functions f in $\operatorname{Lip}(\alpha, p)$ by $t_n(f)$ is as good as the best approximation.

3. AUXILIARY RESULTS

Yano [9] proved that the Walsh–Fejér kernel

$$K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) w_k(t), \qquad n \geqslant 1,$$

is quasi-positive, and $K_{2^m}(t)$ is even positive. These facts are formulated in the following

LEMMA 1. *Let $m \geqslant 0$ and $n \geqslant 1$; then $K_{2^m}(t) \geqslant 0$ for all $t \in I$,*

$$\int_0^1 |K_n(t)| \, dt \leqslant 2 \qquad \text{and} \qquad \int_0^1 K_{2^m}(t) \, dt = 1.$$

A Sidon type inequality proved by Schipp and the author (see [3]) implies that the Nörlund kernel $L_n(t)$ is also quasi-positive. More exactly, $C = [\mathcal{O}(1)]^{1/\gamma} 2\gamma/(\gamma - 1)$ in the next lemma, where $\mathcal{O}(1)$ is from (2.6).

LEMMA 2. *If condition (2.6) is satisfied, then there exists a constant C such that*

$$\int_0^1 |L_n(t)|\, dt \leqslant C, \qquad n \geqslant 1.$$

Now, we give a specific representation of $L_n(t)$, interesting in itself.

LEMMA 3. *Let* $n = 2^m + k$, $1 \leqslant k \leqslant 2^m$, *and* $m \geqslant 1$; *then*

$$\begin{aligned} Q_n L_n(t) = & -\sum_{j=0}^{m-1} r_j(t)\, w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(q_{n-2^{j+1}+i} - q_{n-2^{j+1}+i+1})\, K_i(t) \\ & -\sum_{j=0}^{m-1} r_j(t)\, w_{2^j-1}(t)\, 2^j q_{n-2^j} K_{2^j}(t) \\ & +\sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, D_{2^{j+1}}(t) \\ & + Q_{k+1} D_{2^m}(t) + Q_k r_m(t)\, L_k(t). \end{aligned} \tag{3.1}$$

Proof. The technique applied in the proof is essentially due to Skvorcov [7]. By (2.5),

$$\begin{aligned} Q_n L_n(t) &= \sum_{i=1}^{2^m-1} q_{n-i} D_i(t) + q_{n-2^m} D_{2^m}(t) + \sum_{i=2^m+1}^{2^m+k} q_{n-i} D_i(t) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} q_{n-2^j-i}(D_{2^j+i}(t) - D_{2^{j+1}}(t)) \\ &\quad + \sum_{j=0}^{m-1} \left(\sum_{i=0}^{2^j-1} q_{n-2^j-i} \right) D_{2^{j+1}}(t) \\ &\quad + q_{n-2^m} D_{2^m}(t) + \sum_{i=1}^{k} q_{n-2^m-i} D_{2^m+i}(t). \end{aligned} \tag{3.2}$$

As is well known (see, e.g., [6, p. 46]),

$$D_{2^m+i}(t) = D_{2^m}(t) + r_m(t)\, D_i(t), \qquad 1 \leqslant i \leqslant 2^m. \tag{3.3}$$

Furthermore, by (1.1), it is not difficult to see that

$$w_{2^j-1-l}(t) = w_{2^j-1}(t)\, w_l(t), \qquad 0 \leqslant l < 2^j.$$

Hence we deduce that

$$D_{2^{j+1}}(t)-D_{2^j+i}(t)=r_j(t)\sum_{l=i}^{2^j-1} w_l(t)=r_j(t)\sum_{l=0}^{2^j-i-1} w_{2^j-1-l}(t)$$
$$=r_j(t)\,w_{2^j-1}(t)\,D_{2^j-i}(t),\qquad 0\leqslant i<2^j. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) yields

$$Q_nL_n(t)=-\sum_{j=0}^{m-1} r_j(t)\,w_{2^j-1}(t)\sum_{i=0}^{2^j-1} q_{n-2^j+1}D_{2^j-i}(t)$$
$$+\sum_{j=0}^{m-1}(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\,D_{2^{j+1}}(t)$$
$$+Q_{k+1}D_{2^m}(t)+Q_kr_m(t)\,L_k(t). \tag{3.5}$$

Performing a summation by part gives

$$\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_{2^j-i}(t)$$
$$=\sum_{i=1}^{2^j-1} iK_i(t)(q_{n-2^{j+1}+i}-q_{n-2^{j+1}+i+1})+2^jK_{2^j}(t)\,q_{n-2^j}.$$

Substituting this into (3.5) results in (3.1).

LEMMA 4. *If* $g\in\mathscr{P}_{2^m}$, $f\in L^p$, *where* $m\geqslant 0$ *and* $1\leqslant p\leqslant\infty$, *then for* $1\leqslant p<\infty$

$$\left\{\int_0^1\left|\int_0^1 r_m(t)\,g(t)[f(x\dotplus t)-f(x)]\,dt\right|^p dx\right\}^{1/p}$$
$$\leqslant 2^{-1}\omega_p(f,2^{-m})\int_0^1|g(t)|\,dt, \tag{3.6}$$

while for $p=\infty$

$$\sup\left\{\left|\int_0^1 r_m(t)\,g(t)[f(x\dotplus t)-f(x)]\,dt\!:x\in I\right\}$$
$$\leqslant 2^{-1}\omega_\infty(f,2^{-m})\int_0^1|g(t)|\,dt \tag{3.7}$$

Proof. Since $g\in\mathscr{P}_{2^m}$, it takes a constant value, say $g_m(k)$ on each dyadic interval $I_m(k)$, where $0\leqslant k<2^m$. We observe that if $t\in I_m(k)$ then $t\dotplus 2^{-m-1}\in I_m(k)$.

We will prove (3.6). By Minkowski's inequality in the usual and in the generalized form, we obtain that

$$\left\{\int_0^1 \left|\int_0^1 r_m(t)\, g(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$= \left\{\int_0^1 \left|\sum_{k=0}^{2^m-1} g_m(k) \int_{I_{m+1}(2k)} [f(x \dotplus t) - f(x \dotplus t \dotplus 2^{-m-1})]\, dt\right|^p dx\right\}^{1/p}$$
$$\leqslant \sum_{k=0}^{2^m-1} |g_m(k)| \left\{\int_0^1 \left[\int_{I_{m+1}(2k)} |f(x \dotplus t) - f(x \dotplus t \dotplus 2^{-m-1})|\, dt\right]^p dx\right\}^{1/p}$$
$$\leqslant \sum_{k=0}^{2^m-1} |g_m(k)| \int_{I_{m+1}(2k)} \left\{\int_0^1 |f(x \dotplus t) - f(x \dotplus t \dotplus 2^{-m-1})|^p\, dx\right\}^{1/p} dt$$
$$\leqslant \sum_{k=0}^{2^m-1} |g_m(k)|\, 2^{-m-1} \omega_p(f, 2^{-m}).$$

This is equivalent to (3.6).

Inequality to (3.7) can be proved analogously.

4. Proofs of Theorems 1 and 2

We carry out the *proof of Theorem* 1 for $1 \leqslant p < \infty$. The proof for $p = \infty$ is similar and even simpler.

By (2.4), (3.1), and the usual Minkowski inequality, we may write that

$$Q_n \|t_n(f) - f\|_p := \left\{\int_0^1 \left|\int_0^1 Q_n L_n(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$\leqslant \sum_{j=0}^{m-1} \left\{\int_0^1 \left|\int_0^1 r_j(t)\, g_j(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$+ \sum_{j=0}^{m-1} \left\{\int_0^1 \left|\int_0^1 r_j(t)\, h_j(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$+ \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})$$
$$\times \left\{\int_0^1 \left|\int_0^1 D_{2^{j+1}}(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$+ Q_{k+1} \left\{\int_0^1 \left|\int_0^1 D_{2^m}(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$+ Q_k \left\{\int_0^1 \left|\int_0^1 r_m(t)\, L_k(t)[f(x \dotplus t) - f(x)]\, dt\right|^p dx\right\}^{1/p}$$
$$=: A_{1n} + A_{2n} + A_{3n} + A_{4n} + A_{5n}, \tag{4.1}$$

say, where

$$g_j(t) := w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(q_{n-2^{j+1}+i} - q_{n-2^{j+1}+i+1}) K_i(t),$$

$$h_j(t) := w_{2^j-1}(t)\, 2^j q_{n-2^j} q_{n-2^j} K_{2^j}(t), \qquad 0 \leqslant j < m.$$

Applying Lemma 1, in the case when $q_k \uparrow$ we get that

$$\int_0^1 |g_j(t)|\, dt \leqslant 2 \sum_{i=1}^{2^j-1} i\, |q_{n-2^{j+1}+i} - q_{n-2^{j+1}+i+1}|$$

$$= 2\left(2^j q_{n-2^j} - \sum_{i=1}^{2^j} q_{n-2^{j+1}+i}\right) \leqslant 2^{j+1} q_{n-2^j},$$

while in the case when $q_k \downarrow$

$$\int_0^1 |g_j(t)|\, dt \leqslant 2\left(\sum_{i=1}^{2^j} q_{n-2^{j+1}+i} - 2^j q_{n-2^j}\right)$$

$$\leqslant 2(Q_{n-2^j+1} - Q_{n-2^{j+1}+1}).$$

Thus, by Lemma 4, in the case $q_k \uparrow$

$$A_{1n} \leqslant \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}), \tag{4.2}$$

while in the case $q_k \downarrow$

$$A_{1n} \leqslant \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, \omega_p(f, 2^{-j}). \tag{4.3}$$

By virtue of Lemmas 1 and 4 again, we obtain that

$$A_{2n} \leqslant 2^{-1} \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}). \tag{4.4}$$

Obviously, in the case $q_k \downarrow$

$$2^j q_{n-2^j} \leqslant Q_{n-2^j+1} - Q_{n-2^{j+1}+1}. \tag{4.5}$$

Since

$$D_{2^m}(t) = \begin{cases} 2^m & \text{if } t \in [0, 2^{-m}), \\ 0 & \text{if } t \in [2^{-m}, 1) \end{cases}$$

(see, e.g., [6, p. 7]), by the generalized Minkowski inequality, we find that

$$A_{3n} \leqslant \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) \times \int_0^1 D_{2^{j+1}}(t) \left\{ \int_0^1 |f(x \dotplus t) - f(x)|^p dx \right\}^{1/p} dt$$
$$\leqslant \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) \, \omega_p(f, 2^{-j}), \tag{4.6}$$

$$A_{4n} \leqslant Q_{k+1} \omega(f, 2^{-m}). \tag{4.7}$$

Clearly, in the case $q_k \uparrow$

$$Q_{n-2^j+1} - Q_{n-2^{j+1}+1} \leqslant 2^j q_{n-2^j}. \tag{4.8}$$

Finally, by Lemmas 2 and 4, in a similar way to the above we deduce that

$$A_{5n} \leqslant 2^{-1} Q_k \omega_p(f, 2^{-m}) \int_0^1 |L_k(t)| \, dt \leqslant C Q_n \omega_p(f, 2^{-m}). \tag{4.9}$$

Combining (4.1)–(4.9) yields (2.7) in the case $q_k \uparrow$ and (2.8) in the case $q_k \downarrow$.

Proof of Theorem 2. *Case* (a). $q_k \uparrow$. We have

$$n - 2^j \geqslant 2^{m-1} \qquad \text{for} \quad 0 \leqslant j \leqslant m-1.$$

Consequently, for such j's

$$\frac{2^j q_{n-2^j}}{Q_n} = \frac{(n-2^j+1)\, q_{n-2^j}}{Q_{n-2^j+1}} \frac{Q_{n-2^j+1}}{Q_n} \frac{2^j}{n-2^j+1} \leqslant C 2^{j-m+1},$$

where C equals $\mathcal{O}(1)$ from (2.9). Since $f \in \operatorname{Lip}(\alpha, p)$, from (2.7) it follows that

$$\|t_n(f) - f\|_p = \frac{\mathcal{O}(1)}{Q_n} \sum_{j=0}^{m-1} 2^j q_{n-2^j} 2^{-j\alpha} + \mathcal{O}(2^{-m\alpha})$$
$$= \mathcal{O}(1)\, 2^{-m} \sum_{j=0}^{m} 2^{j-\alpha)}$$
$$= \begin{cases} \mathcal{O}(2^{-m\alpha}) & \text{if} \quad 0 < \alpha < 1, \\ \mathcal{O}(m 2^{-m}) & \text{if} \quad \alpha = 1, \\ \mathcal{O}(2^{-m}) & \text{if} \quad \alpha > 1. \end{cases}$$

This is equivalent to (2.11).

Case (b). $q_k \downarrow$. For example, we consider case (i) in (2.10). Then $Q_n \asymp n^{1-\beta}$. This time we have

$$n - 2^{j+1} \geqslant 2^{m-1} \qquad \text{for} \quad 0 \leqslant j \leqslant m-2.$$

Since $f \in \operatorname{Lip}(\alpha, p)$, from (2.8) it follows that

$$\begin{aligned}
\|t_n(f) - f\|_p &\leqslant \frac{5}{2Q_n} \sum_{j=0}^{m-2} 2^j q_{n-2^{j+1}} \omega_p(f, 2^{-j}) \\
&\quad + \frac{5}{2} \omega_p(f, 2^{-m}) + \mathcal{O}\{\omega_p(f, 2^{-m})\} \\
&= \frac{O(1)}{Q_n} \sum_{j=0}^{m-2} 2^j q_{n-2^{j+1}} 2^{-j\alpha} + \mathcal{O}(2^{-m\alpha}) \\
&= \frac{\mathcal{O}(1)\, 2^{-m\beta}}{n^{1-\beta}} \sum_{j=0}^{m-2} 2^{j(1-\alpha)} + \mathcal{O}(2^{-m\alpha}) \\
&= \begin{cases} \mathcal{O}(n^{-1} 2^{m(1-\alpha)}) & \text{if} \quad 0 < \alpha < 1, \\ \mathcal{O}(n^{-1} m) & \text{if} \quad \alpha = 1, \\ \mathcal{O}(n^{-1}) & \text{if} \quad \alpha > 1. \end{cases}
\end{aligned}$$

Clearly, this is equivalent to (2.11).

Case (ii) in (2.10) can be proved analogously.

5. Concluding Remarks and Problems

(A) We have seen that condition (2.6) is satisfied when $q_k = (k+1)^\beta$ for some $\beta > -1$, and Theorems 1 and 2 apply. If q_k increases faster than a positive power of k, then relation (2.6) is no longer true in general. But the case, for example, when q_k grows exponentially is not interesting, since then condition (2.3) of regularity is not satisfied. On the other hand, the case when $\beta = -1$ is of special interest.

Problem 1. Find substitutes of (2.8) and (2.11) when $q_k = (k+1)^{-1}$. In this case, the $t_n(f)$ are called the logarithmic means for series (2.1).

(B) It is also of interest that Theorems 1 and 2 remain valid when

$$q_k \asymp k^\beta \varphi(k), \tag{5.1}$$

where $\beta > -1$ and $\varphi(k)$ is a positive and monotone (nondecreasing or nonincreasing) functions in k, slowly varying in the sense that

$$\lim_{k \to \infty} \frac{\varphi(2k)}{\varphi(k)} = 1.$$

It is not difficult to check that in this case

$$Q_n \asymp n^{1+\beta}\varphi(n).$$

(C) Now, we turn to the so-called saturation problem concerning the Nörlund means $t_n(f)$. We begin with the observation that the rate of approximation by $t_n(f)$ to functions in $\mathrm{Lip}(\alpha, p)$ cannot be improved too much as α increases beyond 1. Indeed, the following is true.

THEOREM 3. *If $\{q_k\}$ is a sequence of nonnegative numbers such that*

$$\liminf_{m\to\infty} q_{2^m-1} > 0, \tag{5.2}$$

and if for some $f \in L^p$, $1 \leqslant p \leqslant \infty$,

$$\|t_{2^m}(f) - f\|_p = o(Q_{2^m}^{-1}) \quad \text{as} \quad m \to \infty, \tag{5.3}$$

then f is constant.

We note that condition (5.2) is certainly satisfied if $q_k \uparrow$ or $q_k \downarrow$ and $\lim q_k > 0$.

Proof. Since by definition

$$E_{2^m}(f, L^p) \leqslant \|t_{2^m}(f) - f\|_p,$$

and by a theorem of Watari [8]

$$\|s_{2^m}(f) - f\|_p \leqslant 2E_{2^m}(f, L^p),$$

it follows from (5.3) that

$$\|s_{2^m}(f) - f\|_p = o(Q_{2^m}^{-1}) \quad \text{as} \quad m \to \infty. \tag{5.4}$$

A simple computation gives that

$$Q_{2^m}\{s_{2^m}(f, x) - t_{2^m}(f, x)\} = \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^m-k})\, a_k w_k(x).$$

Now, (5.3) and (5.4) imply that

$$\lim_{m\to\infty} \left\| \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^m-k})\, a_k w_k(x) \right\|_p = 0.$$

Since $\|\cdot\|_1 \leqslant \|\cdot\|_p$, for any $p \geqslant 1$ it follows that

$$\lim_{m\to\infty} |(Q_{2^m} - Q_{2^m - j})\, a_j|$$
$$= \lim_{m\to\infty} \left| \int_0^1 w_j(x) \left\{ \sum_{k=1}^{2^m - 1} (Q_{2^m} - Q_{2^m - k})\, a_k w_k(x) \right\} dx \right|$$
$$\leqslant \lim_{m\to\infty} \left\| \sum_{k=1}^{2^m - 1} (Q_{2^m} - Q_{2^m - k})\, a_k w_k(w) \right\|_1 = 0.$$

Hence, by (5.2), we conclude that $a_j = 0$ for all $j \geqslant 1$. Therefore, $f = a_0$ is constant.

In the particular case when $q_k = 1$ for all k, the $t_n(f)$ are the $(C, 1)$-means for series (2.1) defined by

$$\sigma_n(f, x) := \frac{1}{n} \sum_{k=1}^{n} s_k(f, x), \qquad n \geqslant 1,$$

and Theorem 3 is known (see, e.g., [6, p. 191]). It says that if for some $f \in L^p$, $1 \leqslant p \leqslant \infty$,

$$\|\sigma_{2^m}(f) - f\|_p = o(2^{-m}) \qquad \text{as} \quad m \to \infty,$$

then f is necessarily constant.

Problem 2. How can one characterize those functions $f \in L^p$ such that

$$\|\sigma_n(f) - f\|_p = \mathcal{O}(n^{-1}) \qquad \text{for some} \quad 1 \leqslant p \leqslant \infty? \tag{5.5}$$

We conjecture that (5.5) holds if and only if

$$\sum_{m=0}^{\infty} 2^m \omega_p(f, 2^{-m}) < \infty, \qquad \text{or equivalently} \qquad \sum_{k=1}^{\infty} \omega_p(k^{-1}) < \infty.$$

The "if" part can be proved in the same manner as in the case when $\omega_p(f, \delta) = \mathcal{O}(\delta^\alpha)$ for some $\alpha > 1$ (cf. [6, p. 190]). The proof (or disproof) of the "only if" part is a problem.

(D) Finally, we note that the results of this paper can be carried over to the systems that are obtained from the Walsh–Paley system $\{w_k(x)\}$ by means of the so-called piecewise linear rearrangements introduced by Schipp [5]. (See also [7].) In particular, the Walsh–Kaczmarz system is among them.

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